

THE HOROFUNCTION COMPACTIFICATION OF TEICHMÜLLER METRIC

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ABSTRACT. We show that the horofunction compactification of Teichmüller space, endowed with the Teichmüller metric, is homeomorphic to the Gardiner-Masur compactification.

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1. INTRODUCTION

Let $S = S_{g,n}$ be an oriented surface of genus g with n punctures. We assume that $3g - 3 + n > 0$. Denote by $\mathcal{T}(S)$ the Teichmüller space of S and $d_T(\cdot, \cdot)$ the Teichmüller metric. It is well-known that the Teichmüller metric is a complete Finsler metric, and that any two points in Teichmüller space can be connected by a unique geodesic. For $3g - 3 + n = 1$, the Teichmüller metric is isometric to the Poincaré metric on the unit disc, which has constant negative curvature. For $3g - 3 + n > 1$, it was proved by Masur [12] that the Teichmüller metric does not have negative curvature in the sense of Busemann, and it was proved by Masur and Wolf [16] that the Teichmüller metric is not Gromov hyperbolic. For more progress in the study of the Teichmüller metric, we refer to Masur [15] and the references there.

There are several compactifications of $\mathcal{T}(S)$, such as Thurston's compactification, Bers' compactification and the Gardiner-Masur compactification. Compactification of $\mathcal{T}(S)$ is important in the study of mapping class groups and degeneration of quasi-Fuchsian groups (see Thurston [27], Bers [2] and Ohshika [23]). A natural question is to study the relation between the Teichmüller metric and the compactifications of $\mathcal{T}(S)$.

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Teichmüller proved that there is a homeomorphism from the open real $(6g - 6 + 2n)$ -dimensional ball to $\mathcal{T}(S)$, realized by the radial map along Teichmüller geodesic rays from a fixed base-point. Extending the homeomorphism to the closed ball defines a compactification of $\mathcal{T}(S)$, which is called Teichmüller's compactification and denoted by $\overline{\mathcal{T}(S)}^T$.

The *mapping class group* is the group of homotopy classes of orientation-preserving homeomorphisms of S and the *extended mapping class group* is the group of homotopy classes of homeomorphisms of S . Kerckhoff [10] first proved that the action of the mapping class group on $\mathcal{T}(S)$ does not extend continuously to $\overline{\mathcal{T}(S)}^T$. Since the action of the mapping class group on $\mathcal{T}(S)$ extends continuously to Thurston's compactification (Thurston [27], FLP [6]), it follows that Teichmüller's compactification is different from Thurston's compactification. Masur [14] further showed that if a Teichmüller geodesic ray is uniquely ergodic or Strebel, then it converges to a limit point in Thurston's boundary. There exist Teichmüller geodesic rays which do not converge in Thurston's boundary, see Lenzhen [11].

The aim of this chapter is to relate two distinct compactifications of $\mathcal{T}(S)$: one is the horofunction compactification (defined by Gromov [8]) with respect to the Teichmüller metric, and such a compactification could be defined on a quite general class of metric spaces; the other is the Gardiner-Masur compactification, defined by using extremal length of simple closed curves.

The main result in this chapter is:

Theorem 1.1. *The horofunction compactification of $(\mathcal{T}(S), d_T)$ is homeomorphic to the Gardiner-Masur compactification of $\mathcal{T}(S)$.*

The proof of Theorem 1.1 and an explicit homeomorphism from the Gardiner-Masur compactification to the horofunction compactification will be given in Section 5.

There are two corollaries (see Proposition 3.2 and Proposition 3.4 for the proofs):

Corollary 1.2. *The action of the extended mapping class group on $\mathcal{T}(S)$ extends continuously to the Gardiner-Masur boundary.*

Corollary 1.3. *Every Teichmüller (almost-)geodesic ray converges in the forward direction to a point in the Gardiner-Masur boundary.*

Note that Miyachi [18] already gave a different proof of the fact that the action of the mapping class group on $\mathcal{T}(S)$ extends continuously to the Gardiner-Masur boundary.

Miyachi [18], [19] also proved that if a Teichmüller geodesic ray is uniquely ergodic or Strebel, then it converges to a limit point in the Gardiner-Masur boundary. Recall that a Teichmüller geodesic ray is uniquely ergodic or Strebel if the vertical measured foliation of the quadratic differential defining the geodesic ray is uniquely ergodic or Strebel (all of the vertical trajectories are closed). Miyachi [18], [19] proved that if the Teichmüller geodesic ray is uniquely ergodic, the limit point in the Gardiner-Masur boundary is equal to the projective class of the vertical measured foliation (see Theorem 3.1 for the statement); and if the Teichmüller geodesic ray is Strebel, the limit

point is determined by the vertical measured foliation and the conformal structure of the initial point of the geodesic ray.

It follows that the Gardiner-Masur compactification is natural and compatible with the Teichmüller metric, although we do not know much of its geometric structure.

Boundary points of the horofunction compactification are called horofunctions. A horofunction is called a Busemann point if it is a limit point of some almost-geodesic ray (see Section 3 for the definition of almost-geodesic ray). Note that for a Hadamard manifold, i.e., a nonpositively curved and simply connected, the horofunction compactification is the same as the geodesic compactification (see [3]). In particular, each horofunction is a Busemann point.

Since the set of horofunctions of $(\mathcal{T}(S), d_T)$ is identified with the Gardiner-Masur boundary (by Theorem 1.1), it is natural to ask whether every point in the Gardiner-Masur boundary is an accumulation point of a Teichmüller almost-geodesic ray.

We already pointed out that when $3g - 3 + n = 1$, the Teichmüller metric is isometric to the hyperbolic plane. In this case, each horofunction is a Busemann point. When $3g - 3 + n \geq 2$, non-Busemann points of the Gardiner-Masur boundary were recently constructed by Miyachi [20]. He proved that the projective class of a maximal rational measured foliation cannot be the limit of any almost geodesic ray in the Gardiner-Masur compactification.

Inspired by the results of Miyachi, one may ask the following questions:

- Questions 1.4.** (1) Give a necessary and sufficient conditions for a point in the Gardiner-Masur boundary to be a Busemann point. Determine the limit point of a general Teichmüller geodesic ray in the Gardiner-Masur boundary from the conformal structure of its initial point and (the vertical measured foliation of) the quadratic differential determining the geodesic ray.
- (2) Is the set of Busemann points dense in the Gardiner-Masur boundary? Furthermore, understand the geometric structure of the Gardiner-Masur boundary.

Remark 1.5. Recently we learnt from Cormac Walsh that he has solved the first question.

Our proof of Theorem 1.1 is inspired by a recent result of Walsh [28], which appears in this volume. He proved that Thurston's compactification of $\mathcal{T}(S)$ is homeomorphic to the horofunction compactification of $\mathcal{T}(S)$ endowed with Thurston's Lipschitz asymmetric metric. We have felt for some years that the Teichmüller metric is natural for the Gardiner-Masur compactification in some sense, while Thurston's Lipschitz asymmetric metric is natural for Thurston's compactification. Now we know that the horofunction compactification builds a bridge between them.

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2. PRELIMINARIES

The Teichmüller space $\mathcal{T}(S)$ of S is the space of complex structures (or complete, finite-area hyperbolic structures) X on S up to equivalence. We say that two complex structures X and Y are equivalent if there is a conformal map $h : X \rightarrow Y$ homotopic to the identity map on S .

The Teichmüller metric on $\mathcal{T}(S)$ is the metric defined by

$$d_T(X, Y) := \frac{1}{2} \inf_f \log K(f)$$

where $f : X \rightarrow Y$ is a quasi-conformal map homotopic to the identity map of S and

$$\operatorname{ess} - \sup_{x \in X} K_x(f) \geq 1$$

is the quasi-conformal dilatation of f , where

$$K_x(f) = \frac{|f_z(x)| + |f_{\bar{z}}(x)|}{|f_z(x)| - |f_{\bar{z}}(x)|}$$

is the pointwise quasiconformal dilatation at the point $x \in X$ with local conformal coordinate z .

Teichmüller's theorem states that given any $X, Y \in \mathcal{T}(S)$, there exists a unique quasi-conformal map $f : X \rightarrow Y$, called the Teichmüller map, such that

$$d_T(X, Y) = \frac{1}{2} \log K(f).$$

The Beltrami differential $\mu := \frac{\bar{\partial}f}{\partial \bar{f}}$ of the Teichmüller map f is of the form $\mu = k \frac{\bar{q}}{|q|}$ for some quadratic differential q on X and some constant k with $0 \leq k < 1$. In natural coordinates given by q on X and a quadratic differential q' on Y , the Teichmüller map f is given by $f(x + iy) = K^{1/2}x + iK^{-1/2}y$, where $K = K(f) = \frac{1+k}{1-k}$.

The Teichmüller metric is induced by a Finsler norm. Between any two points in $\mathcal{T}(S)$ there is precisely one geodesic. A geodesic ray with initial point X is given by the one-parameter family of Riemann surfaces $\{X_t\}_{t \geq 0}$, where there is a holomorphic quadratic differential q on X and a t -family of Teichmüller maps $f_t : X \rightarrow X_t$, with initial Beltrami differentials $\mu(f_t) = \frac{e^{2t}-1}{e^{2t}+1} \frac{\bar{q}}{|q|}$. Here $\mu(f_t)$ is chosen such that the geodesic ray has arc-length parameter given by t .

Extremal length is an important tool in the study of the Teichmüller metric. The notion is due to Ahlfors and Beurling (ref. [1]). Recall that an *essential* simple closed curve on S is a simple closed curve on S which is neither homotopic to a point on S nor homotopic to a puncture of S . Let \mathcal{S} be the set of homotopy classes of essential simple closed curves on S .

Given a Riemann surface X , a conformal metric σ on X is a metric locally of the form $\sigma(z)|dz|$ where $\sigma(z) \geq 0$ is a Borel measurable function. We define the σ -area of X by

$$A(\sigma) = \int_X \sigma^2(z) |dz|^2.$$

If $\alpha \in \mathcal{S}$, then the σ -length of α is defined by

$$L_\sigma(\alpha) = \inf_{\alpha'} \int_{\alpha'} \sigma(z) |dz|,$$

where the infimum is taken over all essential simple closed curves α' in the homotopy class of α .

With the above notation, we can define the extremal length of α on X by

$$\text{Ext}_X(\alpha) = \sup_{\sigma} \frac{L_\sigma^2(\alpha)}{A(\sigma)},$$

where $\sigma(z)|dz|$ ranges over all conformal metrics on X with $0 < A(\sigma) < \infty$.

The definition of extremal length only depends on the homotopy class of X and the homotopy class of α . Fix $\alpha \in \mathcal{S}$, $\text{Ext}_X(\alpha)$ defines a function on Teichmüller space. The following important formula is due to Kerckhoff [10].

Theorem 2.1. *Let X, Y be any two points of $\mathcal{T}(S)$. Then*

$$d_T(X, Y) = \frac{1}{2} \log \sup_{\alpha \in \mathcal{S}} \frac{\text{Ext}_X(\alpha)}{\text{Ext}_Y(\alpha)}.$$

A *measured foliation* on S is a foliation (with a finite number of singularities) with an invariant transverse measure. The singularities which are allowed are topologically the same as those that occur at $z = 0$ in the line field defined by the quadratic form $z^{p-2}dz^2$. Two measured foliations μ and μ' are *equivalent* if for any simple closed curve γ , the geometric intersection number $i(\gamma, \mu)$ and $i(\gamma, \mu')$ equal. Denote by \mathcal{MF} the space of equivalent classes of measured foliations.

There is a special class of measured foliations that have the property that the complement of the critical leaves is homeomorphic to a cylinder. The leaves of the foliation on the cylinder are then all freely homotopic to a simple closed curve γ . Such a foliation is completely determined as a point in \mathcal{MF} by the height r of the cylinder and the isotopy class of γ . Denote such a foliation by (γ, r) and call it a *weighted simple closed curve*. Thurston [27] showed that \mathcal{MF} is homeomorphic to a $(6g - 6)$ dimensional ball and that there is an embedding $\mathcal{S} \times \mathbb{R}_+ \rightarrow \mathcal{MF}$ whose image is dense in \mathcal{MF} .

By Kerckhoff [10], there is a unique continuous extension of the extremal length function from \mathcal{S} to \mathcal{MF} , with $\text{Ext}_X((\gamma, r)) = r^2 \text{Ext}_X(\gamma)$. As a result, the density of (weighted) simple closed curves in \mathcal{MF} allows us to replace the right hand side of Kerckhoff's formula by the supremum taken over all $\mu \in \mathcal{MF}$:

$$d_T(X, Y) = \frac{1}{2} \log \sup_{\mu \in \mathcal{MF}} \frac{\text{Ext}_X(\mu)}{\text{Ext}_Y(\mu)}.$$

Denote the space of projective measured foliations by \mathcal{PMF} . We may identify \mathcal{PMF} with the cross-section $\{\mu \in \mathcal{MF} \mid \text{Ext}_{X_0}(\mu) = 1\}$ (for any fixed point $X_0 \in \mathcal{T}(S)$) and write Kerckhoff's formula as

$$d_T(X, Y) = \frac{1}{2} \log \sup_{\mu \in \mathcal{PMF}} \frac{\text{Ext}_Y(\mu)}{\text{Ext}_X(\mu)}.$$

Since \mathcal{PMF} is compact, the supremum is attained by some $\mu \in \mathcal{PMF}$.

A measured foliation μ is *minimal* if no closed curve in S can be realized by leaves of μ . Equivalently, after Whitehead moves, the foliation has only dense leaves on S . Two measured foliations μ and ν are *topologically equivalent* if after Whitehead moves, the leaf structures are isotopic to each other. A measured foliation μ is called *uniquely ergodic* if it is minimal and any measured foliation topologically equivalent to μ is measure equivalent to a positive multiple of μ . The following lemma is proved in Masur [14].

Lemma 2.2. *Assume that $\mu \in \mathcal{MF}$ is uniquely ergodic. If $\nu \in \mathcal{MF}$ satisfies $i(\mu, \nu) = 0$, then $\nu = c\mu$ for some constant $c \geq 0$.*

Moreover, it follows from Thurston's theory that uniquely ergodic measured foliations are dense in \mathcal{MF} . This follows from the fact that the orbit of any element of \mathcal{PMF} by the mapping class group action is dense in this space.

3. COMPACTIFICATIONS OF TEICHMÜLLER SPACE

Thurston introduced a compactification $\overline{\mathcal{T}(S)}^{Th}$ of $\mathcal{T}(S)$ such that the action of the mapping class group on $\mathcal{T}(S)$ extends continuously to the boundary $\partial\overline{\mathcal{T}(S)}^{Th}$ of $\overline{\mathcal{T}(S)}^{Th}$. We recall some of the fundamental results of Thurston as described in [6]. Again denote by \mathcal{S} the homotopy classes of essential simple closed curves with the discrete topology. Let $\mathbb{R}_+^{\mathcal{S}}$ be the set of nonnegative functions on \mathcal{S} and let $P(\mathbb{R}_+^{\mathcal{S}})$ be the projective space of $\mathbb{R}_+^{\mathcal{S}}$. Denote by $\pi : \mathbb{R}_+^{\mathcal{S}} \rightarrow P(\mathbb{R}_+^{\mathcal{S}})$ the natural projection. We endow $\mathbb{R}_+^{\mathcal{S}}$ with the product topology and $P(\mathbb{R}_+^{\mathcal{S}})$ with the quotient topology. There is a mapping $\tilde{\psi}$ from $\mathcal{T}(S)$ into $\mathbb{R}_+^{\mathcal{S}}$ which sends X to the function $\tilde{\psi}(X)$ defined by

$$\tilde{\psi}(X)(\alpha) = \ell_X(\alpha)$$

for all $\alpha \in \mathcal{S}$, where $\ell_X(\alpha)$ is the hyperbolic length of α on X . Thurston showed that $\psi = \pi \circ \tilde{\psi} : \mathcal{T}(S) \rightarrow P(\mathbb{R}_+^{\mathcal{S}})$ is an embedding and we call it Thurston's embedding.

There is also an embedding of \mathcal{PMF} into $P(\mathbb{R}_+^{\mathcal{S}})$ which sends each projective class of measured foliation $[\mu]$ to the projective class of the function

$$\gamma \rightarrow i(\mu, \gamma),$$

where $i(\mu, \gamma)$ is the geometric intersection number of the measured foliation μ with the homotopy class of simple closed curve γ . Thurston showed that with these embeddings the closure $\overline{\psi(\mathcal{T}(S))}$ of the image $\psi(\mathcal{T}(S))$ in $P(\mathbb{R}_+^{\mathcal{S}})$ is homeomorphic to a real $(6g - 6 + 2n)$ -dimensional closed ball and \mathcal{PMF} is the boundary sphere of $\overline{\psi(\mathcal{T}(S))}$. We let $\overline{\mathcal{T}(S)}^{Th} = \overline{\psi(\mathcal{T}(S))}$ and call it *Thurston's compactification* of $\mathcal{T}(S)$. The complement $\partial\overline{\mathcal{T}(S)}^{Th} = \overline{\psi(\mathcal{T}(S))} - \psi(\mathcal{T}(S))$ is called *Thurston's boundary* of $\mathcal{T}(S)$. We always identify $\partial\overline{\mathcal{T}(S)}^{Th}$ with \mathcal{PMF} without referring to the embedding.

By replacing the hyperbolic length functions $\ell_X(\alpha)$ by the square root of extremal length functions, Gardiner and Masur [7] defined the *Gardiner-Masur compactification* of $\mathcal{T}(S)$ and the corresponding boundary is called the *Gardiner-Masur boundary*.

Now we give more details. Define a mapping $\tilde{\phi}$ from $\mathcal{T}(S)$ into \mathbb{R}_+^S by

$$\tilde{\phi}(X)(\alpha) = \text{Ext}_X(\alpha)^{1/2}$$

for all $\alpha \in S$. Let $P(\mathbb{R}_+^S)$ be as before the projective space of \mathbb{R}_+^S and $\pi : \mathbb{R}_+^S \rightarrow P(\mathbb{R}_+^S)$ be the natural projection. Gardiner and Masur [7] showed that $\phi = \pi \circ \tilde{\phi} : \mathcal{T}(S) \rightarrow P(\mathbb{R}_+^S)$ is an embedding and the closure $\overline{\phi(\mathcal{T}(S))}$ of the image $\phi(\mathcal{T}(S))$ in $P(\mathbb{R}_+^S)$ is compact. Let $\overline{\mathcal{T}(S)}^{GM} = \overline{\phi(\mathcal{T}(S))}$. Then $\overline{\mathcal{T}(S)}^{GM}$ is called the *Gardiner-Masur compactification* of $\mathcal{T}(S)$. The complement $\partial\overline{\mathcal{T}(S)}^{GM} = \overline{\phi(\mathcal{T}(S))} - \phi(\mathcal{T}(S))$ is called the *Gardiner-Masur boundary* of $\mathcal{T}(S)$.

Gardiner and Masur [7] also proved that $\partial\overline{\mathcal{T}(S)}^{GM Th}$ is a proper subset of $\partial\overline{\mathcal{T}(S)}^{GM}$. For further investigations about Thurston's boundary, the Gardiner-Masur boundary and their relations with the Teichmüller geometry, we refer to Gardiner and Masur [7] and to recent works of Miyachi [18], [19]. We just recall the following theorem that will be used later.

Theorem 3.1 (Miyachi [19]). *Suppose that $r(t)$ is a Teichmüller geodesic ray defined by a quadratic differential q . If the vertical measured foliation $\mathcal{F}_v(q)$ of q is uniquely ergodic, then $r(t)$ converges to a point on the Gardiner-Masur boundary and the limit is equal to the projective class of $\mathcal{F}_v(q)$.*

Miyachi's theorem is an analogue of a theorem of Masur [14] saying that if $\mathcal{F}_v(q)$ is uniquely ergodic, then $r(t)$ converges to a point on Thurston's boundary and that the limit point is equal to the projective class of $\mathcal{F}_v(q)$.

In the following, we will give the definition of horofunction compactification for a general metric space and then use Theorem 1.1 to explain Corollary 1.2 and Corollary 1.3. The proof of Theorem 1.1 will be postponed until Section 5.

Recall that for a locally compact geodesic metric space (M, d) , the horofunction compactification is defined by Gromov [8] in the following way. Fix a base point $x_0 \in M$. To each $z \in M$ we assigned a function $\Psi_z : M \rightarrow \mathbb{R}$, defined by

$$(1) \quad \Psi_z(x) = d(x, z) - d(x_0, z).$$

Let $C(M)$ be the space of continuous functions on M endowed with the topology of locally uniformly convergence on M . Then the map $\Psi : M \rightarrow C(M), z \mapsto \Psi_z$ is an embedding from M into $C(M)$. The closure $\overline{\Psi(M)}$ of $\Psi(M) \subset C(M)$ is compact, and it is called the *horofunction compactification* of (M, d) . The *horofunction boundary* is defined to be

$$M(\infty) = \overline{\Psi(M)} - \Psi(M),$$

and its elements are called *horofunctions*.

Note that here the definition of $M(\infty)$ depends on the choice of the base point x_0 . If one changes to an alternative base point x_1 , then the assignment of the new function Ψ'_z is related to Ψ_z by $\Psi'_z(\cdot) = \Psi_z(\cdot) - \Psi_z(x_1)$. One can check that there is a natural identification between $\overline{\Psi(M)}$ and $\overline{\Psi'(M)}$ and $M(\infty)$ is well-defined up to additive constants.

Let $\text{Isom}(M, d)$ be the isometry group of the metric space (M, d) .

Proposition 3.2. *The action of the isometry group $\text{Isom}(M, d)$ of M extends continuously to a homeomorphism on the horofunction compactification.*

Proof. For any isometry $g \in \text{Isom}(M, d)$ and any horofunction $\xi \in M(\infty)$, we define $g \cdot \xi \in C(M)$ to be

$$(g \cdot \xi)(x) = \xi(g^{-1} \cdot x) - \xi(g^{-1} \cdot x_0).$$

To see that $g \cdot \xi$ is well-defined, assume that $x_n \in M$ converges to ξ , then

$$\begin{aligned} \lim_{n \rightarrow \infty} \Psi_{g \cdot x_n}(x) &= \lim_{n \rightarrow \infty} d(x, g \cdot x_n) - d(x_0, g \cdot x_n) \\ &= \lim_{n \rightarrow \infty} d(g^{-1} \cdot x, x_n) - d(g^{-1} \cdot x_0, x_n) \\ &= \xi(g^{-1} \cdot x) - \xi(g^{-1} \cdot x_0). \end{aligned}$$

It is easy to check that g induces a homeomorphism from the horofunction compactification with base-point x_0 onto the horofunction compactification with base-point $g^{-1} \cdot x_0$. \square

Let $(\mathcal{T}(S), d_T)$ be the Teichmüller space endowed with the Teichmüller metric. Suppose that there is a homeomorphism Ψ between the Gardiner-Masur compactification $\overline{\mathcal{T}(S)}^{GM}$ and the horofunction compactification $\overline{\Psi(\mathcal{T}(S))}$ (This is the statement of Theorem 1.1).

By Proposition 3.2, any element g of the isometry group $\text{Isom}(\mathcal{T}(S), d_T)$ extends continuously to a self-homeomorphism of $\overline{\Psi(\mathcal{T}(S))}$. It is easy to see that the action of g on $\mathcal{T}(S)$ is the same as the action of $\Psi^{-1} \circ g \circ \Psi$. As a result, we have:

Corollary 3.3. *The action of $\text{Isom}(\mathcal{T}(S), d_T)$ on $\mathcal{T}(S)$ extends continuously to a homeomorphism on the Gardiner-Masur compactification.*

Note that the extended mapping class group acts isometrically on the Teichmüller space with the Teichmüller metric. In fact, a famous theorem of Royden (generalized by Earle-Kra [5]) shows that, if S is neither a sphere with ≤ 4 punctures or a torus with ≤ 2 punctures, then the extended mapping class group is precisely the group of isometries of $\mathcal{T}(S)$ with the Teichmüller metric (modulo its center if S is closed of genus 2). Corollary 1.2 follows from Corollary 3.3.

Recall that a *geodesic ray* in a metric space (M, d) is an embedding γ from the interval $[0, \infty)$ to M such that

$$d(\gamma(s), \gamma(t)) = t - s,$$

for all $s, t \in [0, \infty)$, with $s < t$.

A map $\gamma : T \rightarrow M$, where T is an unbounded subset of \mathbb{R}_+ containing 0, is called an *almost-geodesic ray* if for any $\epsilon > 0$ there exists a $m \geq 0$ such that

$$|d(\gamma(0), \gamma(s)) + d(\gamma(s), \gamma(t)) - t| < \epsilon$$

for all $s, t \in T$ with $m \leq s \leq t$. The definition of almost-geodesic ray is due to Rieffel [25].

Rieffel [25] proved the following result:

Proposition 3.4. *Let (M, d) be a metric space. Every almost-geodesic ray of (M, d) converges to a point in $M(\infty)$.*

Now Corollary 1.3, the convergence of Teichmüller almost-geodesic rays to the Gardiner-Masur boundary, follows from Rieffel's result and Theorem 1.1. We learned after finishing this manuscript that Miyachi [20] gave a new proof of Corollary 1.3. Since his proof is short and intrinsic, we state it here for convenience of the readers.

Proof of Corollary 1.3. Let $\gamma : T \rightarrow \mathcal{T}(S)$ be a Teichmüller almost-geodesic ray. For any $\epsilon > 0$, there exists an $m \geq 0$ such that

$$|d_T(\gamma(0), \gamma(s)) + d_T(\gamma(s), \gamma(t)) - t| < \epsilon$$

for all $s, t \in T$ with $m \leq s \leq t$. In particular, taking $s = t \geq m$ in T , we have

$$|d_T(\gamma(0), \gamma(t)) - t| < \epsilon.$$

Let $\log K_{r(t)} = 2d_T(\gamma(0), \gamma(t))$; then the above inequality is equivalent to

$$(2) \quad e^{t-\epsilon} \leq K_{r(t)}^{1/2} \leq e^{t+\epsilon}.$$

For any $m \leq s < t$ in T , by Kerckhoff's distance formula,

$$\sup_{\alpha \in \mathcal{S}} \frac{Ext_{r(t)}^{1/2}(\mu)}{Ext_{r(s)}^{1/2}(\mu)} = e^{d_T(r(s), r(t))} = e^{t-s}.$$

As a result, for any $\alpha \in \mathcal{S}$,

$$\frac{Ext_{r(t)}^{1/2}(\alpha)}{K_{r(t)}^{1/2}} / \frac{Ext_{r(s)}^{1/2}(\alpha)}{K_{r(s)}^{1/2}} = \frac{Ext_{r(t)}^{1/2}(\mu)}{Ext_{r(s)}^{1/2}(\mu)} \cdot \frac{K_{r(s)}^{1/2}}{K_{r(t)}^{1/2}} \leq e^{t-s} \frac{K_{r(s)}^{1/2}}{K_{r(t)}^{1/2}}.$$

By (2), we have

$$\frac{Ext_{r(t)}^{1/2}(\alpha)}{K_{r(t)}^{1/2}} \leq e^{2\epsilon} \frac{Ext_{r(s)}^{1/2}(\alpha)}{K_{r(s)}^{1/2}}.$$

Let $\mathcal{E}_{r(t)}(\alpha) = \frac{Ext_{r(t)}^{1/2}(\alpha)}{K_{r(t)}^{1/2}}$; then

$$(3) \quad \mathcal{E}_{r(t)}(\alpha) \leq e^{2\epsilon} \mathcal{E}_{r(s)}(\alpha).$$

It follows that $\mathcal{E}_{r(t)}(\alpha)$ is a bounded function of t . We set

$$\mathcal{E}(\alpha) = \liminf_{t \in T} \mathcal{E}_{r(t)}(\alpha).$$

Consider the supremum limit $\limsup_{t \in T} \mathcal{E}_{r(t)}(\alpha) < \infty$. By (3), we have

$$\limsup_{t \in T} \mathcal{E}_{r(t)}(\alpha) \leq e^{2\epsilon} \liminf_{t \in T} \mathcal{E}_{r(t)}(\alpha).$$

Since ϵ can be chosen arbitrary small, we have

$$\limsup_{t \in T} \mathcal{E}_{r(t)}(\alpha) = \liminf_{t \in T} \mathcal{E}_{r(t)}(\alpha).$$

As a result, $\mathcal{E}(\alpha)$ is actually the limit of $\mathcal{E}_{r(t)}(\alpha)$.

Note that the above argument is uniformly in $\alpha \in \mathcal{S}$, by the definition of the Gardiner-Masur compactification, $\gamma(t)$ converges to a limit in the Gardiner-Masur boundary. \square

The function $\mathcal{E}_{r(t)}(\alpha)$ is important for understanding the geometry of the Gardiner-Masur compactification. Note that if $r(t)$ is a Teichmüller geodesic ray, then $\mathcal{E}_{r(t)}(\alpha)$ is a decreasing function of t . We will see in next section that this function is generalized by Miyachi [18] to give a representation of each point in the Gardiner-Masur boundary.

4. CONVERGENCE IN THE GARDINER-MASUR COMPACTIFICATION

In the following, when we say that a sequence $P_n \in \overline{\mathcal{T}(S)}^{GM}$ converges to $P \in \overline{\mathcal{T}(S)}^{GM}$, we always refer to the convergence in the sense of the Gardiner-Masur compactification.

Fix a point $X_0 \in \mathcal{T}(S)$ as the base-point of the horofunction compactification. For any $X \in \mathcal{T}(S)$, denote by K_X the dilatation of the Teichmüller map between X_0 and X . Note that $d_T(X_0, X) = \frac{1}{2} \log K_X$ and $K_X = \sup_{\mu \in \mathcal{MF}} \frac{\text{Ext}_X(\mu)^{1/2}}{\text{Ext}_{X_0}(\mu)^{1/2}}$ (here we set the ratio to be 1 when $\mu = 0$). Consider the following function defined on \mathcal{MF} :

$$(4) \quad \mathcal{E}_X(\mu) = \frac{\text{Ext}_X(\mu)^{1/2}}{K_X^{1/2}}, \mu \in \mathcal{MF}.$$

The functions defined in (4) are due to Miyachi [18] and they can be continuously extended to the Gardiner-Masur boundary. They play a role analogous to the intersection numbers $i(\mu, \cdot)$ in Thurston's compactification, as we can see in the following lemma.

Lemma 4.1 (Miyachi [18]). *For any $P \in \partial \overline{\mathcal{T}(S)}^{GM}$, there is a non-negative continuous function $\mathcal{E}_P(\cdot)$ defined on \mathcal{MF} , such that*

(i)

$$\mathcal{E}_P(t\mu) = t\mathcal{E}_P(\mu)$$

for any $t > 0$ and $\mu \in \mathcal{MF}$, and

- (ii) *There is an injective map from $\partial \overline{\mathcal{T}(S)}^{GM}$ to $P(\mathbb{R}_+^{\mathcal{S}})$: each $P \in \partial \overline{\mathcal{T}(S)}^{GM}$ corresponds to the projective class of the function $\mathcal{S} \ni \alpha \mapsto \mathcal{E}_P(\alpha)$.*
- (iii) *The function $\mathcal{E}_P(\cdot)$ is unique up to multiplication by a positive constant in the following sense: for any sequence (X_n) in $\mathcal{T}(S)$ converging to $P \in \overline{\mathcal{T}(S)}^{GM}$, there exists a subsequence (X_{n_j}) such that $\mathcal{E}_{X_{n_j}}(\cdot)$ converges to a positive multiple of $\mathcal{E}_P(\cdot)$ uniformly on any compact subsets of \mathcal{MF} . In particular,*

$$\lim_{n \rightarrow \infty} \frac{\text{Ext}_{X_n}(\mu)^{1/2}}{\text{Ext}_{X_n}(\nu)^{1/2}} = \frac{\mathcal{E}_P(\mu)}{\mathcal{E}_P(\nu)}$$

for all $\mu, \nu \in \mathcal{MF}$ with $\mathcal{E}_P(\nu) \neq 0$.

Miyachi [18] also proved that the projective class of the limit function $\mathcal{E}_P(\cdot)$ is independent of the choice of the base-point X_0 . In the following, if a point P is in $\mathcal{T}(S)$, we will also define $\mathcal{E}_P(\cdot)$ as the function given in (4).

For any $P \in \overline{\mathcal{T}(S)}^{GM}$, we define

$$\mathcal{Q}(P) = \sup_{\nu \in \mathcal{MF}} \frac{\mathcal{E}_P(\nu)}{\text{Ext}_{X_0}(\nu)^{1/2}}$$

and

$$\mathcal{L}_P(\cdot) : \mathcal{MF} \mapsto \mathbb{R}_+ : \mu \rightarrow \frac{\mathcal{E}_P(\mu)}{\mathcal{Q}(P)}.$$

Recall that in (iii) of Lemma 4.1, the function $\mathcal{E}_P(\cdot)$ is defined up to multiplication by a positive constant. Multiplying the function $\mathcal{E}_P(\cdot)$ by a positive constant does not change the value of $\mathcal{L}_P(\cdot)$ and then $\mathcal{L}_P(\cdot)$ is well-defined (independent of the choice of $\mathcal{E}_P(\cdot)$). We may consider $\mathcal{L}_P(\mu)$ as a function of the product space $\overline{\mathcal{T}(S)}^{GM} \times \mathcal{MF}$. Note that Walsh [28] used the geodesic currents theory of Bonahon [4] to define an analogous function on Thurston's compactification, given by

$$(P, \mu) \rightarrow \frac{I_P(\mu)}{\sup_{\nu \in \mathcal{MF}} \frac{I_P(\nu)}{\ell_{X_0}(\nu)}},$$

where $I_P(\nu) = \ell_P(\nu)$ if $P \in \mathcal{T}(S)$ and $I_P(\nu) = i(P, \nu)$ if $P \in \mathcal{PMF} = \partial \overline{\mathcal{T}(S)}^{Th}$.

The importance of $\mathcal{L}_P(\mu)$ is indicated by the following theorem.

Theorem 4.2. *A sequence (P_n) in $\overline{\mathcal{T}(S)}^{GM}$ converges to a point $P \in \overline{\mathcal{T}(S)}^{GM}$ if and only if \mathcal{L}_{P_n} converges to \mathcal{L}_P uniformly on compact set of \mathcal{MF} .*

Proof. (\Rightarrow) First assume that $P \in \mathcal{T}(S)$ and $P_n \rightarrow P$. By definition,

$$\mathcal{Q}(P) = \sup_{\nu \in \mathcal{MF}} \frac{\text{Ext}_P^{1/2}(\nu)}{K_P^{1/2} \text{Ext}_{X_0}(\nu)^{1/2}} = 1.$$

As a result,

$$(5) \quad \mathcal{L}_P(\mu) = \frac{\text{Ext}_P^{1/2}(\mu)}{K_P^{1/2}}.$$

As n is sufficiently large, $P_n \in \mathcal{T}(S)$. The convergence of \mathcal{L}_{P_n} to \mathcal{L}_P follows directly from the continuity of the extremal length function.

Now we assume that $P_n \in \mathcal{T}(S), n = 1, 2, \dots, P \in \partial \overline{\mathcal{T}(S)}^{GM}$ and P_n converges in the Gardiner-Masur compactification to P . Let (P_{n_j}) be any subsequence of (P_n) such that for some $t_0 > 0$, $\mathcal{E}_{P_{n_j}}(\cdot)$ converges to $t_0 \mathcal{E}_P(\cdot)$ uniformly on compact subsets of \mathcal{MF} .

Since

$$\mathcal{Q}(P_n) = \sup_{\nu \in \mathcal{MF}} \frac{\mathcal{E}_{P_n}(\nu)}{\text{Ext}_{X_0}(\nu)^{1/2}} = \sup_{\nu \in \mathcal{PMF}} \frac{\mathcal{E}_{P_n}(\nu)}{\text{Ext}_{X_0}(\nu)^{1/2}},$$

$\mathcal{Q}(P_{n_j})$ converges to $t_0 \mathcal{Q}(P)$. Therefore,

$$\mathcal{L}_{P_{n_j}}(\mu) = \frac{\mathcal{E}_{P_{n_j}}(\mu)}{\mathcal{Q}(P_{n_j})}$$

converges to $\frac{t_0 \mathcal{E}_P(\mu)}{t_0 \mathcal{Q}(P)} = \mathcal{L}_P(\mu)$ uniformly on compact set of \mathcal{MF} . Since the limit is independent of the choice of the subsequence (P_{n_j}) , \mathcal{L}_{P_n} converges to \mathcal{L}_P uniformly on compact sets of \mathcal{MF} .

For the general case, assume that $P_n \in \overline{\mathcal{T}(S)}^{GM}$ and P_n converges in the Gardiner-Masur compactification to $P \in \partial \overline{\mathcal{T}(S)}^{GM}$. It suffices to show that for any fixed compact set K of \mathcal{MF} and for any $\epsilon > 0$, there exists an N , such that for any $n > N$, $|\mathcal{L}_{P_n}(\cdot) - \mathcal{L}_P(\cdot)| < \epsilon$ uniformly on K .

By the above argument, for each P_n , there exists a sequence $(P_{n,k})_{k=1,\dots,\infty}$ in $\mathcal{T}(S)$ such that $\lim_{k \rightarrow \infty} P_{n,k} = P_n$ and $\mathcal{L}_{P_{n,k}}(\cdot)$ converges to $\mathcal{L}_{P_n}(\cdot)$ uniformly on compact sets of \mathcal{MF} .

Set $P'_n = P_{n,n}$. We claim that P'_n converges to P . Otherwise, suppose that there is a subsequence of P'_n , still denoted by P'_n , which converges to some limit $Q \neq P$. We have

(4.1) $\mathcal{L}_P \neq \mathcal{L}_Q$ and then there exists a constant δ_0 and a measured foliation μ_0 such that $|\mathcal{L}_P(\mu_0) - \mathcal{L}_Q(\mu_0)| > \delta_0$.

(4.2) $\mathcal{L}_{P'_n}(\cdot)$ converges to $\mathcal{L}_Q(\cdot)$ uniformly on compact sets of \mathcal{MF} . In particular,

$$|\mathcal{L}_{P'_n}(\mu_0) - \mathcal{L}_Q(\mu_0)| < \delta_0/3$$

for n sufficiently large.

(4.3) By definition of P_n and P'_n ,

$$|\mathcal{L}_{P_n}(\mu_0) - \mathcal{L}_P(\mu_0)| < \delta_0/3,$$

$$|\mathcal{L}_{P'_n}(\mu_0) - \mathcal{L}_{P_n}(\mu_0)| < \delta_0/3$$

for n sufficiently large.

From (4.2), (4.3) and the triangle inequality,

$$|\mathcal{L}_P(\mu_0) - \mathcal{L}_Q(\mu_0)| < \delta_0,$$

which contradicts with (4.1).

As a result, P'_n converges to P . There is a sufficiently large N such that for any $n > N$, we have

$$|\mathcal{L}_{P_n}(\cdot) - \mathcal{L}_{P'_n}(\cdot)| < \frac{\epsilon}{2}, \quad |\mathcal{L}_{P'_n}(\cdot) - \mathcal{L}_P(\cdot)| < \frac{\epsilon}{2}$$

uniformly on K . It follows that $|\mathcal{L}_{P_n}(\cdot) - \mathcal{L}_P(\cdot)| < \epsilon$ uniformly on K .

(\Leftarrow) For $P_n \in \overline{\mathcal{T}(S)}^{GM}$ ($n = 1, 2, \dots$) and P in $\partial \overline{\mathcal{T}(S)}^{GM}$, if \mathcal{L}_{P_n} converges to \mathcal{L}_P uniformly on compact set of \mathcal{MF} , we want to show that P_n converges to P . Let (Y_n) be a subsequence of (P_n) converging in $\overline{\mathcal{T}(S)}^{GM}$ to a point Y . From the above discussion, we have that \mathcal{L}_{Y_n} converges to \mathcal{L}_Y uniformly on any compact set of \mathcal{MF} . Combining this with our assumption that \mathcal{L}_{P_n} converges to \mathcal{L}_P , we have $\mathcal{L}_Y = \mathcal{L}_P$; that is, for any $\mu \in \mathcal{MF}$,

$$\frac{\mathcal{E}_Y(\mu)}{\mathcal{Q}(Y)} = \frac{\mathcal{E}_P(\mu)}{\mathcal{Q}(P)},$$

or equivalently,

$$\mathcal{E}_Y(\mu) = \frac{\sup_{\nu \in \mathcal{MF}} \frac{\mathcal{E}_P(\nu)}{\text{Ext}_{X_0}(\nu)^{1/2}}}{\sup_{\nu \in \mathcal{MF}} \frac{\mathcal{E}_Y(\nu)}{\text{Ext}_{X_0}(\nu)^{1/2}}} \mathcal{E}_P(\mu).$$

Therefore, \mathcal{E}_Y equals to \mathcal{E}_P up to a positive constant. By (ii) of Lemma 4.1, they represent the same point in $\overline{\mathcal{T}(S)}^{GM}$. As we have shown that any convergent subsequence of (P_n) converges to P , it follows that P_n converges to P . \square

Lemma 4.3. *Let (X_n) be a sequence of points in $\mathcal{T}(S)$ that converges to a point P in the Gardiner-Masur boundary. Let Y be a point in $\mathcal{T}(S)$. Let (μ_n) be a sequence in \mathcal{PMF} such that*

$$d_T(X_n, Y) = \frac{1}{2} \log \frac{\text{Ext}_Y(\mu_n)}{\text{Ext}_{X_n}(\mu_n)}.$$

Then any limit point $\mu_\infty \in \mathcal{PMF}$ of a convergent subsequence of the sequence (μ_n) satisfies $\mathcal{E}_P(\mu_\infty) = 0$.

Proof. Since (X_n) converges to P , by (iii) of Lemma 4.1, there exists a subsequence, still denoted by (X_n) , such that $\mathcal{E}_{X_n}(\cdot)$ converges to $t_0 \mathcal{E}_P(\cdot)$ (for some constant $t_0 > 0$) uniformly on compact subsets of \mathcal{MF} .

For any limit point $\mu_\infty \in \mathcal{PMF}$ of a convergent subsequence of the sequence (μ_n) , if $\text{Ext}_P(\mu_\infty) \neq 0$, then the function $\text{Ext}_Y(\mu_n)^{1/2} / \mathcal{E}_{X_n}(\mu_n)$ converges to $\text{Ext}_Y(\mu_\infty)^{1/2} / t_0 \mathcal{E}_P(\mu_\infty)$. On the other hand,

$$\begin{aligned} \frac{\text{Ext}_Y(\mu_n)^{1/2}}{\mathcal{E}_{X_n}(\mu_n)} &= \frac{\text{Ext}_Y(\mu_n)^{1/2}}{\text{Ext}_{X_n}(\mu_n)^{1/2}} K_{X_n}^{1/2} \\ &= e^{d_T(X_n, Y) + d_T(X_n, X_0)} \end{aligned}$$

which tends to ∞ as X_n tends to the boundary.

As a result, $\text{Ext}_Y(\mu_\infty)^{1/2} / t_0 \mathcal{E}_P(\mu_\infty) = \infty$ and then $\mathcal{E}_P(\mu_\infty) = 0$. \square

Lemma 4.4 (Minsky [17]). *For any $\mu, \nu \in \mathcal{MF}(S)$ and any $X \in \mathcal{T}(S)$, we have*

$$i(\mu, \nu) \leq \text{Ext}_X(\mu)^{1/2} \text{Ext}_X(\nu)^{1/2}.$$

Proof. We sketch a simple proof. Note that the intersection number $i(\cdot, \cdot)$ is continuous on $\mathcal{MF}(S) \times \mathcal{MF}(S)$. By the density of simple close curves in $\mathcal{MF}(S)$, it suffices to prove the lemma for any $\alpha, \beta \in \mathcal{S}$. Let q be the one-cylinder Strebel differential of height 1 on X determined by α . By a theorem of Jenkins-Strebel [26], the extremal length $\text{Ext}_X(\alpha)$ is realized by:

$$\text{Ext}_X(\alpha) = \frac{L_\rho^2(\alpha)}{A(\rho)},$$

where $\rho = |q|^{1/2} |dz|$.

The complement of vertical critical leaves is a cylinder foliated by circles isotopic to α . Note that the circumference and height of the cylinder are equal to $L_\rho(\alpha)$ and 1 respectively. As a result, $L_\rho(\alpha) = A(\rho)$ and $\text{Ext}_X(\alpha) = L_\rho(\gamma_1) = A(\rho)$.

Since the length of β measured by the metric ρ is larger than $i(\alpha, \beta)$, we have a lower bound of $\text{Ext}_X(\beta)$:

$$\text{Ext}_X(\beta) \geq \frac{i(\alpha, \beta)^2}{A(\rho)} = \frac{i(\alpha, \beta)^2}{\text{Ext}_X(\alpha)}.$$

As a result, $\text{Ext}_X(\alpha)^{1/2} \text{Ext}_X(\beta)^{1/2} \geq i(\alpha, \beta)$. \square

Lemma 4.5. *Let (X_n) be a sequence of points in $\mathcal{T}(S)$ that converges to a point P in the Gardiner-Masur boundary. Let Y be a point in $\mathcal{T}(S)$. Let (μ_n) be a sequence in \mathcal{PMF} such that*

$$d_T(X_n, Y) = \frac{1}{2} \log \frac{\text{Ext}_Y(\mu_n)}{\text{Ext}_{X_n}(\mu_n)}.$$

For any $\nu \in \mathcal{MF}$, if $\mathcal{E}_P(\nu) = 0$, then any limit point $\mu_\infty \in \mathcal{PMF}$ of a convergent subsequence of the sequence (μ_n) satisfies $i(\nu, \mu_\infty) = 0$.

Proof. Consider any subsequence of (X_n) , still denoted by (X_n) , such that $\mathcal{E}_{X_n}(\cdot)$ converges to $t_0 \mathcal{E}_P(\cdot)$ (for some constant $t_0 > 0$) uniformly on any compact subset of \mathcal{MF} .

By Lemma 4.4, we have

$$i(\nu, \mu_n) \leq \text{Ext}_{X_n}(\nu)^{1/2} \text{Ext}_{X_n}(\mu_n)^{1/2}.$$

Note that

$$\begin{aligned} & \text{Ext}_{X_n}(\nu)^{1/2} \text{Ext}_{X_n}(\mu_n)^{1/2} \\ &= K_{X_n}^{1/2} \frac{\text{Ext}_{X_n}(\nu)^{1/2}}{K_{X_n}^{1/2}} \frac{\text{Ext}_{X_n}(\mu_n)^{1/2}}{\text{Ext}_Y(\mu_n)^{1/2}} \text{Ext}_Y(\mu_n)^{1/2} \\ &= e^{d_T(X_0, X_n)} \mathcal{E}_{X_n}(\nu) \frac{1}{e^{d_T(Y, X_n)}} \text{Ext}_Y(\mu_n)^{1/2} \\ &\leq \sup_{\mu \in \mathcal{PMF}} \{\text{Ext}_Y(\mu)^{1/2}\} \mathcal{E}_{X_n}(\nu) e^{d_T(X_0, X_n) - d_T(Y, X_n)} \\ &\leq C e^{d_T(X_0, Y)} \mathcal{E}_{X_n}(\nu), \end{aligned}$$

where C is a constant depending on Y . Taking a limit, we have

$$i(\nu, \mu_\infty) \leq C t_0 e^{d_T(X_0, Y)} \mathcal{E}_P(\nu).$$

As a result, if $\mathcal{E}_P(\nu) = 0$, then $i(\nu, \mu_\infty) = 0$. □

From the above lemmas, we have

Corollary 4.6. *Let (X_n) be a sequence of points in $\mathcal{T}(S)$ that converges to a point P in the Gardiner-Masur boundary. Let Y be a point in $\mathcal{T}(S)$. Let (μ_n) be a sequence in \mathcal{PMF} such that*

$$d_T(X_n, Y) = \frac{1}{2} \log \frac{\text{Ext}_Y(\mu_n)}{\text{Ext}_{X_n}(\mu_n)}.$$

Let $\mu_\infty, \mu'_\infty \in \mathcal{PMF}$ be limit points of two different convergent subsequences of the sequence (μ_n) . Then $i(\mu_\infty, \mu'_\infty) = 0$.

5. PROOF OF THEOREM 1.1

Our proof of Theorem 1.1 is inspired by Walsh [28]. He showed that Thurston's compactification $\overline{\mathcal{T}(S)}^{Th}$ of $\mathcal{T}(S)$ is homeomorphic to the horofunction compactification of $\mathcal{T}(S)$ with Thurston's Lipschitz metric. Moreover, for each $\mu \in \mathcal{PMF} = \partial \overline{\mathcal{T}(S)}^{Th}$ (Thurston's boundary), the corresponding horofunction, which we denote by Ψ_μ^{Th} , is given by

$$(6) \quad \Psi_\mu^{Th}(X) = \log \sup_{\nu \in \mathcal{MF}} \frac{i(\mu, \nu)}{\ell_X(\nu)} - \log \sup_{\nu \in \mathcal{MF}} \frac{i(\mu, \nu)}{\ell_{X_0}(\nu)},$$

where X_0 is a fixed base point in $\mathcal{T}(S)$.

To prove Theorem 1.1, we will construct the horofunctions of $\mathcal{T}(S)$ with the Teichmüller metric by replacing the hyperbolic length $\ell_X(\mu)$ with the square root of the extremal length $\text{Ext}_X(\mu)^{1/2}$ and the intersection number $i(\mu, \cdot)$ by the function $\mathcal{E}_\mu(\cdot)$ (defined in Section 4).

For each $P \in \overline{\mathcal{T}(S)}^{GM}$, we define the map

$$\begin{aligned}\Psi_P(X) &= \log \sup_{\mu \in \mathcal{MF}} \frac{\mathcal{E}_P(\mu)}{\text{Ext}_X(\mu)^{1/2}} - \log \sup_{\mu \in \mathcal{MF}} \frac{\mathcal{E}_P(\mu)}{\text{Ext}_{X_0}(\mu)^{1/2}} \\ &= \log \sup_{\mu \in \mathcal{MF}} \frac{\mathcal{L}_P(\mu)}{\text{Ext}_X(\mu)^{1/2}}\end{aligned}$$

for all $X \in \mathcal{T}(S)$. The last equality follows by the definition

$$\mathcal{L}_P(\mu) = \mathcal{E}_P(\mu) / \sup_{\mu \in \mathcal{MF}} \frac{\mathcal{E}_P(\mu)}{\text{Ext}_{X_0}(\mu)^{1/2}}.$$

Note that if $P \in \mathcal{T}(S)$, by (4) and Kerckhoff's formula,

$$\begin{aligned}\Psi_P(X) &= \log \sup_{\mu \in \mathcal{MF}} \frac{\text{Ext}_P(\mu)^{1/2}}{K_P^{1/2} \text{Ext}_X(\mu)^{1/2}} - \log \sup_{\mu \in \mathcal{MF}} \frac{\text{Ext}_P(\mu)^{1/2}}{K_P^{1/2} \text{Ext}_{X_0}(\mu)^{1/2}} \\ &= \log \sup_{\mu \in \mathcal{MF}} \frac{\text{Ext}_P(\mu)^{1/2}}{\text{Ext}_X(\mu)^{1/2}} - \log \sup_{\mu \in \mathcal{MF}} \frac{\text{Ext}_P(\mu)^{1/2}}{\text{Ext}_{X_0}(\mu)^{1/2}} \\ &= d_T(X, P) - d_T(X_0, P).\end{aligned}$$

Thus, the function Ψ_P coincides with the function defined in (1) for the case where the metric space is $(\mathcal{T}(S), d_T)$. We will show that it is injective and continuous. Then Theorem 1.1 will follow from a topological argument.

Proposition 5.1. *The map $\Psi : \overline{\mathcal{T}(S)}^{GM} \rightarrow C(\mathcal{T}(S)) : P \rightarrow \Psi_P$ is injective.*

Proof. To prove the assertion, it suffices to prove that for any two distinct points $P, Q \in \overline{\mathcal{T}(S)}^{GM}$, there exists a point $X \in \mathcal{T}(S)$ such that $\Psi_P(X) \neq \Psi_Q(X)$.

By Theorem 4.2, \mathcal{L}_P and \mathcal{L}_Q are distinct. Without loss of generality, we assume that $\mathcal{L}_P(\mu) < \mathcal{L}_Q(\mu)$ for some $\mu \in \mathcal{PMF}$. Since \mathcal{L}_P and \mathcal{L}_Q are continuous, we can take a neighborhood \mathcal{N} of μ in \mathcal{PMF} and real numbers u and v such that

$$(7) \quad \mathcal{L}_P(\nu) \leq A < B \leq \mathcal{L}_Q(\nu).$$

for all $\nu \in \mathcal{N}$.

We recall that $\mathcal{PMF} = \partial \mathcal{T}(S)^{Th} \subset \partial \mathcal{T}(S)^{GM}$. Since the set of uniquely ergodic measured foliations is dense in \mathcal{PMF} , we can choose a uniquely ergodic measured foliation $\mu_0 \in \mathcal{N}$. By identifying \mathcal{PMF} with a subset of the Gardiner-Masur boundary, we also choose μ_0 such that $P \neq \mu_0$. By Theorem 3 in [19], this condition of μ_0 means that $\mathcal{L}_P(\mu_0) \neq 0$. Therefore, there is an $M_1 > 0$ such that $\mathcal{L}_P(\mu_0) \geq M_1 \mathcal{L}_P(\nu)$ for any $\nu \in \mathcal{PMF}$.

Let $r(t)$ be a Teichmüller geodesic ray defined by a quadratic differential q on X_0 with uniquely ergodic vertical measured foliation $\mathcal{F}_v(q) = \mu_0$. By Theorem 3.1, $r(t)$ converges to μ_0 (considered as a point in the Gardiner-Masur boundary), with $\mathcal{E}_{\mu_0}(\cdot) = i(\mu_0, \cdot)$ up to a positive multiplicative constant.

From the sharpness of Minsky's inequality (see Theorem 5.1 in [7]), we have

$$\sup_{\nu \in \mathcal{MF}} \frac{i(\mu_0, \nu)}{\text{Ext}_{X_0}^{1/2}(\nu)} = \text{Ext}_{X_0}^{1/2}(\mu_0) = 1.$$

It follows that $\mathcal{L}_{r(t)}(\cdot)$ converges to $\mathcal{L}_{\mu_0}(\cdot)$ uniformly on compact sets of \mathcal{MF} and

$$\mathcal{L}_{\mu_0}(\cdot) = i(\mu_0, \cdot).$$

We claim that for t sufficiently large, the supremum of $\mathcal{L}_P(\cdot)/\text{Ext}_{r(t)}^{1/2}(\cdot)$ is attained in the set \mathcal{N} . To see this, note that there are $T_0 > 0$ and $0 < M_2 < 1$ such that $\mathcal{L}_{r(t)}(\nu) > M_2$ for all $\nu \in \mathcal{PMF} \setminus \mathcal{N}$ but $\mathcal{L}_{r(t)}(\mu_0) < M_1 M_2$ for $t \geq T_0$. Therefore,

$$\begin{aligned} \frac{\mathcal{L}_P(\mu_0)}{\text{Ext}_{r(t)}^{1/2}(\mu_0)} &= K_{r(t)}^{1/2} \frac{\mathcal{L}_P(\mu_0)}{\mathcal{L}_{r(t)}(\mu_0)} \\ &> K_{r(t)}^{1/2} \frac{\mathcal{L}_P(\mu_0)}{M_1 M_2} \\ &\geq K_{r(t)}^{1/2} \frac{\mathcal{L}_P(\nu)}{M_2} \\ &\geq K_{r(t)}^{1/2} \frac{\mathcal{L}_P(\nu)}{\mathcal{L}_{r(t)}(\nu)} \\ &= \frac{\mathcal{L}_P(\nu)}{\text{Ext}_{r(t)}^{1/2}(\nu)} \end{aligned}$$

for all $\nu \in \mathcal{PMF} \setminus \mathcal{N}$.

As a result,

$$\begin{aligned} \sup_{\mu \in \mathcal{PMF}} \frac{\mathcal{L}_P(\mu)}{\text{Ext}_X^{1/2}(\mu)} &= \sup_{\mu \in \mathcal{N}} \frac{\mathcal{L}_P(\mu)}{\text{Ext}_X^{1/2}(\mu)} \\ &\leq \sup_{\mu \in \mathcal{N}} \frac{A}{\text{Ext}_X^{1/2}(\mu)} \\ &< \sup_{\mu \in \mathcal{N}} \frac{B}{\text{Ext}_X^{1/2}(\mu)} \\ &\leq \sup_{\mu \in \mathcal{PMF}} \frac{\mathcal{L}_Q(\mu)}{\text{Ext}_X^{1/2}(\mu)}. \end{aligned}$$

Thus $\Psi_P(X) < \Psi_Q(X)$. □

The following topological lemma will be used later. The proof here is given by Walsh [28].

Lemma 5.2. *Let X and Y be two topological spaces and let $\Psi : X \times Y \rightarrow \mathbb{R}$ be a continuous function. Let (x_n) be a sequence in X converging to $x \in X$. Then, $\Psi(x_n, \cdot)$ converges to $\Psi(x, \cdot)$ uniformly on compact sets of Y .*

Proof. Let K be any compact subset of Y . For any $\epsilon > 0$ and $y \in Y$, there are open neighborhoods $x \in U_y \subset X, y \in V_y \subset Y$, such that

$$|\Psi(x', y') - \Psi(x, y)| < \epsilon, \quad \forall x' \in U_y, y' \in V_y.$$

Since K is compact, it can be covered by a finite collection of such neighborhoods $\{V_{y_1}, \dots, V_{y_n}\}$. Set $U = \cap_{i=1, \dots, n} U_{y_i}$. Then for any $x \in U$, $|\Psi(x', y') - \Psi(x, y)| < \epsilon$ holds uniformly for all $y \in K$. \square

Lemma 5.3. *The map $\Psi : \overline{\mathcal{T}(S)}^{GM} \rightarrow C(\mathcal{T}(S)) : P \rightarrow \Psi_P$ is continuous.*

Proof. Let P_n be a sequence of $\overline{\mathcal{T}(S)}^{GM}$ converging to a point P_0 in $\overline{\mathcal{T}(S)}^{GM}$. By Theorem 4.2, \mathcal{L}_{P_n} converges to \mathcal{L}_{P_0} uniformly on compact sets of \mathcal{MF} . For all $X \in \mathcal{T}(S)$, the square root of the extremal function $\text{Ext}_X^{1/2}$ is bounded away from zero on \mathcal{PMF} . We conclude that for any $X \in \mathcal{T}(S)$, $\frac{\mathcal{L}_{P_n}}{\text{Ext}_X^{1/2}}$ converges uniformly on \mathcal{PMF} to $\frac{\mathcal{L}_{P_0}}{\text{Ext}_X^{1/2}}$. Since

$$\Psi_P(X) = \log \sup_{\mu \in \mathcal{PMF}} \frac{\mathcal{L}_P(\mu)}{\text{Ext}_X(\mu)^{1/2}},$$

Ψ_{P_n} converges pointwise to Ψ_{P_0} . Since the function $\Psi : \overline{\mathcal{T}(S)}^{GM} \times \mathcal{T}(S) \rightarrow \mathbb{R} : (P, X) \rightarrow \Psi_P(X)$ is continuous, and P_n converges to P_0 in $\overline{\mathcal{T}(S)}^{GM}$, by Lemma 5.2, $\Psi_{P_n}(\cdot)$ converges to $\Psi_{P_0}(\cdot)$ uniformly on any compact set of $\mathcal{T}(S)$. By the definition of the topology of $C(\mathcal{T}(S))$, the map $\Psi : P \rightarrow \Psi_P(\cdot)$ is continuous. \square

Theorem 5.4. *The map Ψ is a homeomorphism between the horofunction compactification of $\mathcal{T}(S)$ with the Teichmüller metric and the Gardiner-Masur compactification of $\mathcal{T}(S)$.*

Proof. We have shown that $\Psi : \overline{\mathcal{T}(S)}^{GM} \rightarrow C(\mathcal{T}(S))$ is injective and continuous. Note that an embedding from a compact space to a Hausdorff space must be a homeomorphism to its image (see Kelley [9], Page 141 for the proof). As a result, $\Psi(\overline{\mathcal{T}(S)}^{GM})$ is a compact subset of $C(\mathcal{T}(S))$. Since the horofunction compactification is the closure of $\Psi(\mathcal{T}(S))$, it is equal to $\Psi(\overline{\mathcal{T}(S)}^{GM})$. \square

Miyachi [18] constructed an embedding Φ from $\overline{\mathcal{T}(S)}^{GM}$ into the space $C(\mathcal{MF}(S))_{\geq 0}$ of non-negative continuous functions on $\mathcal{MF}(S)$, defined by

$$\Phi_P(\cdot) := \frac{\mathcal{E}_P(\cdot)}{\mathcal{E}_P(\alpha) + \mathcal{E}_P(\beta)}$$

where α, β is a pair of simple closed curves filling S . Such an embedding allows us to give a distance function on $\overline{\mathcal{T}(S)}^{GM}$ by

$$\text{dist}(P, Q) = \max_{\mu \in \mathcal{PMF}} |\Phi_P(\mu) - \Phi_Q(\mu)|.$$

The topology induced by the distance is compatible with the topology given by the embedding $\Psi(\overline{\mathcal{T}(S)}^{GM}) \subseteq C(\mathcal{T}(S))$.

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